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Dirac's bra-ket notation and the notion of quantum state[1]

The relationship between physical notions and the mathematical structures expressing them is an issue of primary importance in the history and philosophy of pre-modern natural philosophy and of modern science. Historical evidence suggests that, when investigating the relationship between physics and mathematics, one should pay attention to the material, sensually perceivable forms mediating it: spoken or written words, drawings, artefacts, descriptions of experimental set-ups or, as in the case discussed in the present paper, symbolic notation. The reason for this is that, historically, physical notions and their mathematical expressions have very often been developed as one, and only distinguished from each other at a later stage. This was for example the case with what we today regard as "mathematical analysis" and "classical mechanics", which were developed in the course of the eighteenth century as the discipline of the mathematical analysis of motion.

When considered from a later point of view, both the physical notions and the mathematical concepts involved in historical developments appear as being "in flux", "vague", and, as far as the mathematical side is concerned, "non-rigorous". In such cases, verbal description, mechanical artefacts or symbols do not simply mediate between physical notions and mathematical structures, as might be the case at a later stage, but represent both of them in one, allowing the historical actors to construct and manipulate physical-mathematical meaning by following, at times, their physical reasoning and, at times, their criteria of mathematical rigour.

The history of quantum mechanics offers many examples of how new physics and new mathematics can emerge as one, with symbolic notation playing a central role in shaping both. Among the many creators of quantum theory, Paul Dirac was particularly sensitive to questions of notation:

> [...] a good notation can be of great value in helping the development of a theory, by making it easy to write down those quantities or combinations of quantities that are important, and difficult or impossible to write down those that are unimportant. (Dirac, 1939 p. 416)

So Dirac wrote in a paper published in 1939, which he devoted exclusively to introducing "A new notation for quantum mechanics". The new formalism, the bra-ket notation, was the final result of a long development which I shall sketch in the following pages, as I believe it offers a very good example of the combined evolution of (1) the physical notion of a state of a quantum system, (2) the mathematical structure employed to represent it and (3) the visual and in some sense haptic features of the notation mediating between the two.

1. Bohr's stationary states and their representations

In the early twentieth century, a main problem facing physicists trying to explain the structure of matter according to classical mechanics and electromagnetism was the stability of

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atomic systems. Experimental evidence strongly suggested that atoms contained moving charges, and thus, according to classical theory, would have been expected to continuously radiate electromagnetic waves and lose energy (the following overview on the development of quantum mechanics is based on: Jammer (1966)). Instead, atoms were stable and only emitted (or absorbed) radiation under specific circumstances. Moreover, each kind of atom only emitted and absorbed electromagnetic waves of specific frequencies.

To explain these phenomena, Niels Bohr in 1913 postulated that atomic systems obeyed a non-classical dynamics according to which they would be most of the time in 'stationary states' that, per definition, did not radiate. Each stationary state was characterized by a specific, constant energy, which could only take one of a discrete set of values. These values were ultimately derived from spectroscopic evidence, and were expressed in terms of one or more integers, the 'quantum numbers'. Radiation occurred only when an atom passed from one stationary state to another, with a net loss or gain in energy. According to Bohr's theory, atoms did not change state smoothly, as classical systems did, but rather 'jumped' from one stationary state to the other, emitting or absorbing the energy difference ΔE in the form of light with a specific frequency v determined by Planck's relation $\Delta E = hv$, where h was Planck's quantum of action.

To obtain predictions for the frequencies and intensities of emitted and absorbed radiation, stationary states were represented using classical mechanical formalism, e.g. in terms of orbits. However, atomic orbits were assumed to satisfy specific 'quantisation conditions' imposed onto the classical formalism from the outside. Thus, in the old quantum theory, no closed, self-contained mathematical theory of stationary states existed: some elements had to be put in by hand, on the ground of spectroscopic data. Following Bohr's approach, various quantum theories were formulated in the second and third decade of the twentieth century, but no agreement could be reached as to how far classical physical-mathematical notions such as orbits could be assumed to actually describe the physical features of quantum systems. In fact, not even general agreement on the choice of quantum numbers to label the energy levels existed. The notions of stationary states and quantum jumps, however, remained central to the theory in all its different forms.

2. Stationary states and the new quantum mechanics

Between 1925 and 1926, a series of fundamental breakthroughs were realized. In 1925, Werner Heisenberg, Max Born and Pascual Jordan proposed a new dynamics ('matrix mechanics') in which physical notions such as 'position', 'velocity' or 'energy' were not represented anymore by a single numerical value, but rather by an infinite set of numbers: infinite-dimensional matrices whose rows and columns were labelled by the quantum numbers of the old quantum theory. Thus, Bohr's stationary states came to formally correspond to the rows and columns of infinite matrices. Energy was represented in this scheme by a diagonal matrix whose generic non-zero element E_{nn} was equal to the energy of the n-th stationary state. In matrix mechanics, matrices corresponding to the coordinates of a system did not commute with those linked to the corresponding momenta, and this condition formally prevented from assigning values to both coordinates and momenta at the same time. Thus, matrix mechanics hardly helped clarify the notion of stationary state, but rather reinforced the impression that atomic systems could not be described in classical terms. Matrix mechanics was also problematical from the mathematical point of view, since its authors offered no rigorous definition of infinite-dimensional matrices and their properties.

At the beginning of 1926, Erwin Schrödinger proposed a new theory, 'wave mechanics', whose key element was the equation now known as Schrödinger's equation. The novelty of this equation was that its solutions could be classified in terms of integer numbers having the same values as the quantum numbers of the hydrogen atom of Bohr's theory. Similar equations could be written also for quantum systems more complex than the hydrogen atom and, although in those cases the equation could not be explicitly solved, it was assumed that the results obtained for the hydrogen atoms could in principle be generalized to all quantum systems. Therefore, in wave mechanics, stationary states could be described by a mathematical formalism which predicted the values of the quantum numbers and was not dependent on externally imposed quantum conditions.

Schrödinger believed that the new mathematical representation of the stationary states in terms of the functions solving his equation would provide the starting point for a new physical interpretation of atomic systems, with the wave function $\psi(x)$ interpreted as representing a really existing 'matter wave'. The notion of quantum jumps should eventually be replaced by some continuous formalism.

During the second half of 1926, matrix mechanics and wave mechanics merged into a new quantum mechanics. At the same time, Schrödinger's wave interpretation of quantum mechanics appeared increasingly problematical. Eventually, Max Born tentatively proposed to interpret the square $|\psi(x)|^2$ of the wave function as the probability of finding the relevant quantum system in position x. This kind of interpretation was fruitful for the further development of the theory, but did little to clarify whether and how one might physically define and mathematically describe the state of a quantum system. In fact, in this period, the term 'state' ('Zustand') continued being used only to denote the energy levels of atoms, i.e. Bohr's stationary states.

3. Dirac's transformation theory of 1927

In late 1926 and in 1927, physicists struggled to come to terms with the many implications of the new quantum mechanics and proposed expansions and reformulations of its complex formal machinery which would highlight its physical significance. After Born had made the first step in the direction of a probabilistic interpretation, others followed, attempting to develop a formalism capable of deriving probability distributions not only for position and energy, as born had already done, but also for all other possible observable quantities of a system. Two researchers devoted particular efforts to generalizing quantum theory in this direction: Pascual Jordan and Paul Dirac. In the following pages, I will discuss Dirac's work, where he introduced a new notation which, as we shall see, would develop into the bra-ket formalism (for a detailed discussion of Dirac's work, see Kragh (1990), Darrigol (1992)).

In January 1927, Dirac published a paper on "The physical interpretation of the quantum dynamics"(Dirac, 1927). As others before him had done, he noted that quantum mechanics provided less information on the evolution of a system than classical mechanics, since the values of coordinates q and momenta p could not be determined both at the same time (Dirac, 1927 p. 623). However, some questions could be answered also in the quantum case:

> One cannot answer any questions on the quantum theory which refers to numerical values for both the q_{r0} and the p_{r0}. One would expect, however, to be able to answer questions in which only the q_{r0} or only the p_{r0} are given numerical values, or, more generally, when any set of constants of integration ξ_r that commute with one another are given numerical values. (Dirac, 1927 p. 623)

Dirac noted here that 'the words constant of integration include such quantities as the value of a varying co-ordinate or momentum at a specified time' (Dirac, 1927 p. 623). So, the quantities ξ_r could be any dynamical variables of the system.. Dirac assumed the quantities ξ_r to be a set of generalized coordinates which, together with the corresponding generalized momenta η_r, would have completely described the system in classical mechanics. The formalism of quantum mechanics allowed the values of such a set of generalized coordinates to be determined at the same time, and Dirac accordingly assumed the numerical values ξ_r' of the quantities ξ_r to be given. What more could one learn about the system, and how?

> To answer questions in which the ξ_r are given numerical values, we require a scheme of matrices to represent the dynamical variables, whose rows an columns refer to numerical values for the ξ_r. [...] We therefore require a theory of the more general scheme of matrix representations, in which the rows and columns refer to any set of constants of integration that commute, and of the laws of transformation from one such scheme to another. (Dirac, 1927 p. 623-624)

Here, Dirac was assigning to a thoroughly generic set of coordinates the same formal role that the quantum numbers of stationary states had had until then. It is easy, almost too easy, for today's readers to implicitly understand Dirac's 'scheme' in term of matrices having rows and columns corresponding to generic quantum states labelled by the quantum numbers ξ_r'. As I will show later on, though, there is plenty of evidence that Dirac did not consider the set of ξ_r as describing a quantum state.

4. Dirac's transformation functions and their notation (1927)

Dirac had set himself the task of finding rules allowing to take the matrix which was associated to a physical quantity g and had rows and columns labelled by the values of the quantities α_r, and transform it into a matrix still associated with g, but having rows and columns labelled by another set of quantities ξ_r. In the course of the paper, Dirac developed the formulas necessary to answer this question and, with each step, he introduced a new notation that would best fit his purpose: sets of variables like ξ_r should be written simply as ξ, and the relevant numerical values as primed letters ξ', ξ'', while the matrices representing the same physical quantity g in two different schemes were $g(\xi'\ \xi'')$ and $g(\alpha'\ \alpha'')$ (Dirac, 1927 p. 629–630). Moreover, with his usual nonchalance with respect to mathematical rigour, Dirac explained that, in the general case, ξ and α varied in a continuous way, so that in fact the 'matrices' presented a continuous range of 'rows' and 'columns'.

Notheless, Dirac continued to refer to $g(\xi'\ \xi'')$ and $g(\alpha'\ \alpha'')$ as matrices and speak of their rows and columns. Taking the unproblematic theory of finite matrices as a model-pattern, he wrote down the transformation formula between two matrix schemes in the form:

$$g(\xi'\ \xi'') = \int\int (\xi' / \alpha')\ d\ \alpha'\ g(\alpha'\ \alpha'')\ (\alpha''/\xi'')\ d\ \alpha'' \quad \text{(Dirac, 1927 p. 630)}$$

The readers familiar with matrix algebra will immediately recognize that, in this expression, the indices α' and α'' were being 'summed over' and substituted by the indices ξ' and ξ'', as one would do in a multiplication between finite matrices. However, Dirac never spoke of his transformations as changes of base, as he might have done (Dirac, 1927 p. 628–631).

The functions determining the transformation were indicated by the symbols (ξ'/α') and (α''/ξ''), which Dirac introduced here for the first time. These 'transformation functions' connected matrices representing the same physical variable, but having different labels for their rows and columns. With this notation, the indices for rows and columns of a matrix had

achieved symbolic independence from the specific matrices they labelled: the transformation from α to ξ always followed the pattern defined by the function (ξ'/α'), regardless of the values of the matrix elements. In the new notation, the labels for rows and columns even became independent from each other, since one could write:

$$g(\xi' \alpha') = \int (\xi' / \alpha'') \, d\alpha'' \, g(\alpha'' \alpha') \quad \text{(Dirac, 1927, p. 630)}$$

Dirac noted the novelty of these hybrid entities, which he however continued to indicate with the increasingly questionable term 'matrix' (Dirac, 1927 p. 630). With his notation, he had opened up a whole new space of physical-mathematical operations and of potential physical-mathematical meaning. He had done so by means of symbolic manipulations, without ever referring either to relevant mathematical structures (e.g. a change of base), or to physical notions like a generic quantum state.

Dirac now derived from quantum mechanics both the formal properties and the physical meaning of the new hybrid entities: (ξ'/g') was the complex conjugate of (g'/ξ'), and their square modulus $|(\xi'/g')|^2$ gave the probability distribution for the value ξ' of the variable ξ when the variable g took the value g' – and viceversa. The average value of g when the values ξ' were given was the diagonal matrix element $g(\xi' \xi')$. This was the answer to the initial question of the paper (Dirac, 1927 p. 637–638). Dirac then applied his new theory to electron-atom scattering processes:

> To determine [the scattering] probability by the present method, we must find the transformation function (p_F'/p_I') that connects the final component of momentum of the electron p_F, with the initial components p_I.. There is then a probability $(p_I'/p_F')\ dp_F'\ (p_F'/p_I') = |(p_F'/p_I')|^2\ dp_F'$ that the electron will be scattered with a momentum lying in the range dp_F'. (Dirac, 1927 p. 640)

Again, it is almost unavoidable for today's reader to think of the two halves of the transformation function as individuating respectively the 'initial state' and 'final state' of the scattering process, but this was not the way in which Dirac saw things at that time. In the 1927 paper, Dirac used the term 'state' either to refer to atomic stationary states, or, surprisingly, to claim that quantum states could in principle be described by assigning the values of both coordinates and momenta at the same time (Dirac, 1927 p. 622, 639, 641). Indeed, in this paper Dirac attempted to interpret quantum probability distributions in terms of traditional statistical ensembles (Dirac, 1927 p. 641). It is not here the place to discuss whether and how he might have been trying to formulate some kind of 'hidden variables' model, but one thing is certain: at this stage, Dirac did not regard the assignation of the values for a maximium set of commuting observables ξ as the most general definition of the state of a quantum system.

Finally, it must be noted that the mathematical methods which Dirac employed in this paper contained both questionable and definitively non-rigorous aspects, as for example the informal transition from discrete to continuous matrices and the insufficiently defined differentiation and integration procedures. Despite the physical and mathematical vagueness, though, Dirac's transformation theory possessed a clear, easy to manipulate structure and greatly contributed to shape the further development of quantum mechanics.

5. Von Neumann's Hilbert space and the new notion of quantum state (1927/28)

While these developments were going on, mathematicians became involved in the effort of straightening out both the mathematics and the physics of quantum mechanics. A particularly important role was played by John von Neumann, whose two papers on the foundations of

quantum mechanics were published in the 1927 issue of the ‘Göttinger Nachrichten’, printed in 1928 (von Neumann 1928a, 1928b). In the first paper, on "The mathematical foundation of quantum mechanics", von Neumann analysed the various formulations of quantum mechanics (wave mechanics, matrix mechanics, transformation theory), isolated their common structural features and then offered a rigorous definition of mathematical structures possessing exactly those features: these would be the mathematical foundations of quantum mechanics.

This strategy led him to define a new mathematical object: the abstract Hilbert space, whose elements ('vectors') were neither numbers nor functions, but could be added and multiplied following a set of axioms which, as one might expect, closely resembled the way in which wave functions could be added and multiplied. von Neumann also provided axiomatic definitions for linear operators acting on Hilbert vectors: these operators could be manipulated like infinite matrices, but were rigorously defined. In this first paper, von Neumann did not claim that abstract Hilbert vectors would represent the states of a quantum system: as all other authors before him, he spoke of 'states' ('Zustände') only to indicate systems which had a specific value of the energy (von Neumann, 1928a p. 47–50). The notion of a quantum state was introduced only in the second paper in the series: ’Probabilistic-theoretical structure of quantum mechanics’ (von Neumann, 1928b).

In this complex and insightful paper, von Neumann addressed the statistical, mathematical and operational implications of the fact that, for quantum systems, one could not in principle obtain as much information through measurement as in the classical case. He gave a formal definition of when a quantum system could be regarded as being in a 'completely determined state' ('vollständing bestimmter Zustand', von Neumann, 1928b p. 219). It would be impossible for me to summarize here his analysis, which set the foundation for later discussions of the measurement problem and of entanglement. For the purposes of this paper it is sufficient to note that, according to von Neumann, a quantum system should be regarded in a completely determined state when the values of a maximum set of commuting, independent operators were known. von Neumann referred to such completely determined states simply as 'states' (Zustände), and showed that they were represented by vectors in abstract Hilbert space, although some vectors could correspond to the same state, i.e. to identical values of the same physical quantities (von Neumann, 1928b p. 258.) Hilbert vectors were indicated simply with letters, e.g. *f*, *g*, etc, evetually with index label to denote their properties (e.g. energy values). The scalar product of two vercotrs *f* and *g* was always a number and took the form *Q(f,g)*.

Thus, vectors in abstract Hilbert space represented a new physical notion of quantum states, and operators acting on them stood for physical variables. In hindsight, Dirac's symbol $g(\xi' \xi')$ could thus be interpreted as the average value of *g* in a quantum state defined by the values ξ' . The transformation function (g' / ξ') corresponded in von Neumann’s theory to the scalar product of two state vectors $\varphi_{g'}$ and $\varphi_{\xi'}$. However, to express what for Dirac were transoformation functions, von Neumann had used not only a completely different formalism, but also a quite complex mathematical construction: the 'resolution of the identity' ('Zerteilung der Einheit'), which I shall not discuss here (von Neumann, 1927a p. 50–51).

6. The first edition of Dirac's "Principles of quantum mechanics" (1930)

Paul Dirac's "The Principles of quantum mechanics" was one of the most influential first-generation textbooks on quantum theory (Brown 2006). It was first published in 1930, and later updated by the author in a second (1935), third (1947) and fourth (1958) edition, the first three differing each from the previous one in the presentation of the subject matter. The bra-

ket notation was used for the first time in the 1947 edition, after having been introduced in the already mentioned paper from 1939. However, already in the preface to the first edition Dirac addressed the question of the form in which to present quantum theory:

> With regard to the mathematical form in which the theory can be presented, an author must decide at the outset between two methods. There is the symbolic method, which deals directly in an abstract way with the quantities of fundamental importance [...] and there is the method of the coordinates or representations, which deals with sets of numbers corresponding to these quantities. (Dirac, 1930 p. ii)

Of the two mathematical forms Dirac mentioned, the symbolic method, as we shall see, was his own version of von Neumann's approach; the other one was, of course, the older formalism of matrices and wave functions. Dirac chose the symbolic road:

> The symbolic method [...] seems to go more deeply into the nature of things. It enables one to express the physical laws in a neat and concise way, and will probably be increasingly used in the future as it becomes better understood and its own special mathematics gets developed. (Dirac, 1930 p. ii)

As I will try to show, the choice Dirac made here was not primarily a decision between two different mathematical structures (e.g. abstract Hilbert space vs. functions and matrices), but rather between two kinds of notation that were visually perceived and mentally manipulated in very different ways. According to him, the symbolic method offered patterns of manipulation reflecting more closely physical ideas. In the first edition of the "Principles", the physical notion of "quantum state" and of its peculiar non-classical features, which had been absent form the 1927 paper, played a prominent role. The notion of state was introduced using photon polarization as an example, and then generalized:

> We must first generalize the meaning of a 'state' so that it can apply to any atomic system. Corresponding to the case of the photon, which we say is in a given state of polarization when it has passed through suitable polarizing apparatus, we say that any atomic system is in a given state when it has been prepared in a given way, which may be repeated arbitrarily at will. The method of preparation may then be taken as the specification of the state. (Dirac, 1930 p. 7)

Thus, Dirac gave what we today would regard as an operational definition of a quantum state. The second chapter of the book was devoted to the 'Symbolic algebra of states and observables', and began with the words:

> We introduce certain symbols which we say denote physical things such as states of a system or dynamical variables. These symbols we shall use in algebraic analysis in accordance with certain axioms which will be laid down. To complete the theory we require laws by which any physical conditions may be expressed by equations between the symbols and by which, conversely, physical results may be inferred from equations between the symbols. (Dirac, 1930 p. 18)

When reading this passage, one is tempted to interpret the 'axioms' in the sense of von Neumann's paper, i.e. as mathematical axioms rigorously defining specific algebraic structures which could be manipulated in (essentially) the same way as the non-rigorous quantities of Dirac's transformation theory. However, this was not what Dirac had in mind: his 'axioms' simply assigned the rules according to which the symbols could be manipulated

without regard for mathematical rigour. The symbols were at this point not supposed to correspond to any specific mathematical structure:

> One does not anywhere specify the exact nature of the symbols employed, nor is such specification at all necessary. They are used all the time in an abstract way, the algebraic axioms that they satisfy and the connection between equations involving them and physical conditions being all that is required. (Dirac, 1930 p. 18)

Later on, however, Dirac did mention that his physical-mathematical formalism might be understood in terms of vectors. These passages highlight Dirac's approach to formalism, particularly his notions of 'symbol' and 'symbolic formalism', which defied an immediate identification both with physical notions and with rigorously defined mathematical entities. In chapter five, Dirac introduced his transformation theory: transformation functions were now written as $(r \mid s)$ and were shown to be identical to the scalar product of two states. In the second edition of the "Principles" (1935), the notation $(r \mid s)$, was given a more prominent role: it was introduced already in chapter three, was used to express the wave functions and matrices, and was even given a proper name: the 'bracket notation' (Dirac, 1935 p. 49–52).

7. Dirac's bra-ket notation (1939)

The importance attributed by Dirac to questions of symbolic notation was confirmed in 1939, when he devoted a paper to the introduction of "A new notation for quantum mechanics". In this paper, Dirac took as a starting point the "two styles of calculation in two distinct notations" of quantum mechanics, which he had already discussed in the preface to the first edition of the "Principles". Now, he explicitly stated that he considered the existence of two notations unsatisfactory not only for general reasons of coherence, but also because it disturbed a physicist's reasoning:

> For the two styles of calculation, two distinct notations are used, which do not fit together very naturally and which give rise to an awkward jump in the flow of one's thoughts when one changes from one to the other. In the present note a new notation is set up, which provides a neat and concise way of writing, in a single scheme, both the abstract quantities themselves and their coordinates, and thus leads to a unification of ideas. (Dirac, 1939 p. 416).

Here we see clearly how Dirac's new notation was supposed to mediate not between physical ideas and mathematical structures, but rather between two different sets of mathematical structures (Hilbert vectors and operators vs. function/matrices) and physical notions (quantum states and physical variables vs. average values and probabilities), thus leading to a "unification of ideas" which one might assume to be physically significant, although Dirac here did not discuss this matter at all. In introducing the new symbols, Dirac referred explicitly to abstract Hilbert spaces:

> A Hilbert space vector, which was denoted in the old notation by the letter ψ, will now be denoted by a special new symbol $> .$ If we are concerned with a particular vector, specified by a label, a say, which would be used as a suffix to the ψ in the old notation, we write it $\mid a >$. It may be that the label is very complicated, consisting of many letters, but we can always write down the vector conveniently in the new notation, simply by enclosing the label between $\mid$ on the left hand and $>$ on the right. (Dirac, 1939 p. 416).

As had already happened in the 1927 paper, here, too, labels took centre stage. In the expression | a >, the state itself was represented by a bar and a parenthesis, i.e. a sort of meta-symbol, which conveyed the message that the letter *a* did not simply represent the value of a physical variable, but a feature individuating a quantum state. In this notation, the state was an empty variable to be filled with quantum numbers, a feature of the bra-ket notation which would play a very important role later on. The new notation allowed to bypass the highly nontrivial question that one could not, in a rigorous formulation of abstract Hilbert space, simply take the product of two vectors to obtain a number: to do this, the notion of complex conjugate vector had to be introduced. In Dirac's new notation, a very easy symbolic trick allowed to bypass this problem in a visually appealing way:

> We have also to deal with another kind of Hilbert-space vector, the conjugate imaginary of the first kind. This was denoted in the old notation by φ or by [ψ with a bar on top], and will now be denoted by <. If one of them is specified by a label a, we write it < *a* |. (Dirac, 1939 p. 416)

The new symbols < *a* | and | *b* > conveyed not only a visual, but also a haptic message: they invited physicist to fit them to each other, in a product < *a* | *b* > which turned out to be a number: 'any quantity in brackets < > is a number, and any expression containing an unclosed bracket symbol < or > is a vector in Hilbert space' (Dirac, 1939 p. 418). As names for the new symbols 'to be used in speech', Dirac proposed respectively 'bra' and 'ket', thus expressing in common language the same invitation to put together the two halves in order to obtain a familiar object: a bracket (Dirac, 1939 p. 418). Other kinds of products of bras and kets were possible: | *a* > | *b* >, < *a* |< *b* | ,| *a* >< *b* |, and each of them had its specific physical-mathematical meaning, even though mathematicians would for a long time regard the notation as not rigorous (Jauch, 1972).

As was to be expected, the transformation functions indicated in 1927 as (*a* / *b*) and in 1930 as (*a* | *b*) now took the form < *a* | *b* > and could be interpreted physically, mathematically and symbolically as the product of the state vectors < *a* | and | *b* > . In this way, Dirac had achieved the unification of the two 'mathematical forms' between which he had had to choose in 1930: the bra-ket notation allowed to pass from one to the other without any 'awkward jump in the flow of one's thoughts ' (Dirac, 1939 p. 418).

In the third edition of the "Principles" (1947), Dirac not only employed the bra-ket notation, but also introduced it already in the first chapter, immediately after the mathematical notion of vector had been associated to the physical idea of a quantum state (Dirac, 1947 p. 15–22). The popularity of the "Principles" made the bra-ket notation well known to physicists, although it was rarely used to actually perform computations. With the development of quantum field theory and, later on, quantum optics and quantum computing, the notation came to be often used in cases in which no representation of physical quantities in terms of wave functions or matrices was possible, but where one could still somehow extract predictions from the theory. One example for all will have to suffice here: theoretical physicists almost always employ the bra-ket notation when they want to formally represent the vacuum state of a quantum field: 0 | > , a ket with a zero as a label (e.g. Itzykson/Zuber, 1980 p. 512–513). The vacuum state is both physically and mathematically one of the most important and problematical notions of quantum field theories, and it might be described colloquially as a state of particles which are not there: an empty variable. Still today, the bra-ket notation helps when both rigorous mathematics and physical intuition reach their limits.

8. Summary and conclusions

Looking back at the development sketched in the previous sections, one might summarize it as follows: in 1913, Niels Bohr introduced the radically innovative notion of a stationary state in which atoms did not radiate. This idea of non-classical states of quantum systems continued being employed in early quantum mechanics, where it was associated to the rows and columns of infinite-dimensional matrices, as well as to some specific solutions of Schrödinger's equation. In 1927, building upon earlier versions of quantum mechanics, Dirac introduced new patterns of manipulation of quantum matrices and wave functions which greatly contributed to the further development of quantum mechanics, although they were mathematically questionable and were apparently linked to a non-viable interpretation of the theory in traditional statistical terms. In this context, Dirac also introduced a very user-friendly symbolism for associating the rows and columns of quantum matrices to the values of any generic set of commuting variables of a quantum system. However, he did not show any intention of using this formalism as a means of modifying the notion of a quantum state.

Shortly afterwards, von Neumann distilled out of the various contemporary versions of quantum mechanics some mathematically rigorous structures, which he subsequently employed for investigating the question of how to physically define and mathematically describe the possible states of a quantum system and the knowledge that one might obtain of them through measurement. Among the results of this approach were a new mathematical structure (abstract Hilbert space) and a generalized notion of quantum states, which was defined by the values of a maximum set of physical quantities which could all be determined at the same time.

Von Neumann had not simply effected a separation between mathematical structures and physical notions, but had constructed both of them out of the physical-mathematical material at his disposition. This statement should not be understood as saying that von Neumann created quantum mechanics: he only gave his own contribution, as many before had done. He expanded, sharpened and, in some aspects, modified the theory, particularly as far as probabilistic issues were concerned. Eventually, Dirac came to appreciate those aspects of von Neumann's work where the manipulation patterns of his own transformation theory were presented in their most abstract form. However, Dirac was not ready to burden his own view of quantum mechanics with von Neumann's detailed axiomatic approach. By developing the bra-ket notation, he managed to blur some of the sharp boundaries drawn by von Neumann and other mathematicians, and put out of focus some foundationally relevant but practically cumbersome details. In this way, Dirac could extract some elements of the new mathematical structures ('vectors in a certain space') and physical notions (state of a quantum system) and fold them back onto the flowing physical-mathematical formalism which he had introduced in 1927 and whose patterns – rather unsurprisingly – well fitted later developments.

Thanks to the bra-ket notation, the visual and haptic appeal of transformation theory was restored in enhanced form to quantum mechanics. Dirac's students and, in general, all the readers of the "Principles" were now invited to learn and interiorise quantum dynamics in a way in which the English physicist believed 'went more deeply into the nature of things' (Dirac, 1930 p. ii). This was, perhaps, Dirac's way of addressing foundational issues, on which he otherwise remained remarkably silent. It would make little sense to ask whether it was Dirac, von Neumann or someone else to be the first to reach more deeply into the nature of quantum dynamics by defining and describing a generic quantum state. Hopefully, in telling the story of the bra-ket notation, I have succeeded in my aim of showing how important the choice of specific formal symbols and rules of manipulation was in this context,

and that no historical or philosophical analysis can allow itself to reduce the discourse to a mere interaction of physical notions and mathematical structures, failing to acknowledge the role of the mediating instances: symbolic notations.

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